1969

Homotropy: The Fundamental Group

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TITLE OF THESIS

HOMOTOPY: THE FUNDAMENTAL GROUP

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July 31 '69

(date)

Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Arts.

Northern Michigan University
Marquette, Michigan

July, 1969
(date)

HOMOTOPY: THE FUNDAMENTAL GROUP

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This paper originated from work in algebraic topology in the course M.A. 515 at Northern Michigan University. The paper introduces the homotopy group and produces the group for the spaces S (the unit circle in two space) and $R^3 - S$. It assumes the reader has a basic knowledge of group theory and elementary point-set topology.

a continuous function $\emptyset$: X x I $\rightarrow$ Y such that $\emptyset(x, 0) = f(x)$ and $\emptyset(x, 1) = g(x)$ for all x in X. Define $\emptyset'$:X x I $\rightarrow$ Y by $\emptyset'(x, t) = \emptyset(x, 1-t)$. Then $\emptyset'(x, 0) = \emptyset(x, 1) = g(x)$ and $\emptyset'(x, 1) = \emptyset(x, 0) = f(x)$ for all x in X. Since the mapping t into 1-t is continuous, $\emptyset'$ is continuous by composition of continuous functions. Thus $\emptyset'$ is sufficient to establish that $g \approx f$. If $f \approx_a g$, $\emptyset(a, t) = f(a) = g(a)$ for all t in I so $\emptyset'(a, t) = \emptyset(a, 1-t) = f(a) = g(a)$ for all t in I. Hence $g \approx_a f$.

(3) Suppose $f \approx g$ and $g \approx h$. By definition of homotopic, there exist continuous functions $\emptyset_1$, $\emptyset_2$:X x I $\rightarrow$ Y such that $\emptyset_1(x, 0) = f(x)$, $\emptyset_1(x, 1) = g(x)$ and $\emptyset_2(x, 0) = g(x)$, $\emptyset_2(x, 1) = h(x)$ for all x in X. Define $\emptyset$: X x I $\rightarrow$ Y by $\emptyset(x, t) = \emptyset_1(x, 2t)$ for $0 \leq t \leq 1/2$ and $\emptyset(x, t) = \emptyset_2(x, 2t-1)$ for $1/2 \leq t \leq 1$. Since the mapping t into 2t is continuous $\emptyset$ is continuous on [0, 1/2] and since the mapping t into 2t-1 is continuous $\emptyset$ is continuous on [1/2, 1]. When t = 1/2, $\emptyset_1(x, 2t) = \emptyset_1(x, 1) = g(x) = \emptyset_2(x, 0) = \emptyset_2(x, 2t-1)$. Therefore, $\emptyset$ is continuous. Also $\emptyset(x, 0) = \emptyset_1(x, 0) = f(x)$ and $\emptyset(x, 1) = \emptyset_2(x, 1) = h(x)$. Hence $\emptyset$ is sufficient to establish that $f \approx h$. If $f \approx_a g$ and $g \approx_a h$, $\emptyset_1(a, t) = f(a) = g(a)$ and $\emptyset_2(a, t) = g(a) = h(a)$ for all t in I. Hence $\emptyset(a, t) = f(a) = h(a)$ for all t in I. Therefore, $f \approx_a h$.

DEFINITION TWO: Let $a \in X$, X a topological space. Let $C_a(X) = \{f: I \rightarrow X \text{ is continuous and } f(o) = f(1) = a\}$

DEFINITION THREE: Define $f * g$: I $\rightarrow$ X by $(f * g)(t) = f(2t)$ for $0 \leq t \leq 1/2$ and $(f * g)(t) = g(2t-1)$ for $1/2 \leq t \leq 1$, f, g in $C_a(X)$

THEOREM TWO: Let X be a topological space and f', f", g', g" in $C_a(X)$ for some a in X. Then $f' \approx_{0,1} f"$ and $g' \approx_{0,1} g"$ implies $f' * g' \approx_{0,1} f" * g"$.

Proof: If $f' \approx_{0,1} f"$ and $g' \approx_{0,1} g"$, there exist continuous functions $\emptyset_f$, $\emptyset_g$: I x I $\rightarrow$ X such that $\emptyset_f(x, 0) = f'(x)$, $\emptyset_f(x, 1) = f"(x)$, $\emptyset_f(0, t) = \emptyset_f(1, t) = a$, $\emptyset_g(x, 0) = g'(x)$ $\emptyset_g(x, 1) = g"(x)$ and $\emptyset_g(0, t) = \emptyset_g(1, t) = a$ for all x in X and t in I. Define $\emptyset(x, t) = \emptyset_f(2x, t)$ for $0 \leq x \leq 1/2$ and $\emptyset(x, t) = \emptyset_g(2t-1, t)$ for $1/2 \leq x \leq 1$. Then $\emptyset(1/2, t) = \emptyset_f(1, t)$ $= a = \emptyset_g(0, t) = \emptyset(1/2, t)$ for all t in I. Since Ø is well defined at t = 1/2 and continuous on I x [0, 1/2] and I x [1/2, 1] Ø is continuous on I.

When t = 0 $\emptyset(x, 0) = \begin{cases} \emptyset_f(2x, 0) = f'(2x) & 0 \leq x \leq 1/2 \\ \emptyset_g(2x-1, 0) = g'(2x-1) & 1/2 \leq x \leq 1 \end{cases}$

$= f' * g' (x)$

When t = 1 $\emptyset(x, 1) = \begin{cases} \emptyset_f(2x, 1) = f"(2x) & 0 \leq x \leq 1/2 \\ \emptyset_g(2x-1, 1) = g"(2x-1) & 1/2 \leq x \leq 1 \end{cases}$

$= f" * g" (x)$

$\emptyset(0, t) = \emptyset_f(0, t) = a = \emptyset_g(1, t) = \emptyset(1, t)$ for all t in I.

Hence $f' * g' \approx_{0,1} f" * g"$

DEFINITION FOUR: Let $\pi(X, a) = C_a(X)/\approx_{0,1}$. If $\overline{f}$ denotes the equivalence class of f, define $\overline{f} * \overline{g} = \overline{f * g}$ for $\overline{f}$, $\overline{g}$ in $C_a(X)/\approx_{0,1}$. $\pi(X, a)$ is called the fundamental group of X modulo a.

THEOREM THREE[2]: $\pi(X, a)$ is a group.

Proof: Consider $\pi(X, a)$ with operation "*" as defined in definition three. To show that $\pi(X, a)$ is a group with respect to "*" we must show:

(1) that "*" is well-defined and $\pi(X, a)$ is closed under "*",

(2) that "*" is associative between homotopy classes,

(3) that there exists an identity element, and

(4) that each homotopy class has an inverse.

(1) To show that "*" is well-defined we must show that we get the same equivalence class $f * g$ no matter which f and g we select from their respective equivalence classes. But by theorem two, if $f' \approx_{0,1} f''$ and $g' \approx_{0,1} g''$, then $f' * g' \approx_{0,1} f'' * g''$ and by definition four, $\overline{f' * g'} = \overline{f'} * \overline{g'}$. Therefore, "*" is well-defined. $\pi(X,a)$ is also closed under "*" since for $\overline{f}$, $\overline{g}$ in $C_a(X)/\approx_{0,1} = \pi(X, a)$ with $\overline{f}$ the equivalence class of f and $\overline{g}$ the equivalence class of g, then $\overline{f} * \overline{g} = \overline{f * g}$, the equivalence class of $\overline{f * g} \in C_a(X)/\approx_{0,1} = \pi(X, a)$

(2) Let $f, g, h \in C_a(X)$ and $\overline{f}$, $\overline{g}$, $\overline{h}$ the respective homotopy equivalence classes. We must show that $f * (g * h) \approx_{0,1} (f * g) * h$. Since f, g and h are all continuous functions, $f * g$, $g * h$, $(f * g) * h$, and $f * (g * h)$ are each continuous. Applying the operation "*"

$$(f * g) * h\ (t) = \begin{cases} (f * g)\ (2t) & 0 \leq t \leq 1/2 \\ h(2t-1) & 1/2 \leq t \leq 1 \end{cases} = \begin{cases} f(4t) & 0 \leq t \leq 1/4 \\ g(4t-1) & 1/4 \leq t \leq 1/2 \\ h(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

$$f * (g * h)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ (g * h)(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

$$= \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(4t - 2) & 1/2 \leq t \leq 3/4 \\ h(4t - 3) & 3/4 \leq t \leq 1 \end{cases}$$

We are now ready to show $(f * g) * h \approx f * (g * h)$. Define $\emptyset: I \times I \longrightarrow X$ by:

$$\emptyset(t, s) = \begin{cases} f(4t/1 + s) & s \geq 4t - 1 \\ g(4t - s - 1) & 4t - 1 \geq s \geq 4t - 2 \\ h(4t - s - 2/2-t) & 4t - 2 \geq s. \end{cases}$$

Then $\emptyset(t, 0) = \begin{cases} f(4t) & 0 \geq 4t - 1 \quad \text{or } 0 \leq t \leq 1/4 \\ g(4t-1) & 4t - 1 \geq 0 \geq 4t - 2 \quad \text{or } 1/4 \leq t \leq 1/2 \\ h(2t-1) & 4t - 2 \geq 0 \quad \text{or } 1/2 \leq t \leq 1 \end{cases}$

$= (f * g) * h$

and $\emptyset(t, 1) = \begin{cases} f(2t) & 1 \geq 4t - 1 \quad \text{or } 0 \leq t \leq 1/2 \\ g(4t-2) & 4t - 1 \geq 1 \geq 4t-2 \quad \text{or } 1/2 \leq t \leq 3/4 \\ h(4t-3) & 4t - 2 \geq 1 \quad \text{or } 3/4 \leq t \leq 1 \end{cases}$

$= f * (g * h)$

If $s = 4t - 1$, $\emptyset(t, s) = f(1) = g(0) = a$ and if $s = 4t - 1$ $\emptyset(t, s) = g(1) = h(0) = a$. Therefore $\emptyset$ is properly defined at the endpoints of thre three trapezoids and is the composition of continuous functions on each trapezoid. Hence $\emptyset$ is continuous and is sufficient to show the desired homotopy.

(3) To show that an identity exists, let $k(t) = a$ for each t in I, k a constant mapping from I to X. For f in $C_a(X)$ define $\emptyset: I \times I \longrightarrow X$ by $\emptyset(t, s) = f(2t/ 1 + s)$ for $s \geq 2t - 1$ and $\emptyset(t, s) = a$ for $s \leq 2t - 1$.

$g(2t-2/s-1)$ since $1 \notin 2t-1$ when $1/2 \leq t \leq 1$. Again, $\emptyset$ is either a composition of continuous functions or a constant and is properly defined everywhere, so $\emptyset$ is continuous everywhere. Thus $\overline{g'}$ is the inverse under "*" of $\overline{g}$.

THEOREM FOUR[3]: If a, b in X, a topological space, and there exists a function $f: I \longrightarrow X$ such that a, b in f(I), then $\pi(X, a)$ is isomorphic to $\pi(X, b)$.

Proof: Let h be a continuous mapping of I into X such that $h(0) = a$ and $h(1) = b$ and let f be any continuous mapping of I into X such that $f(0) = f(1) = a$.

$$\text{Define } \emptyset(f)(s) = \begin{cases} \overline{h}(1-3s) & 0 \leq s \leq 1/3 \\ f(3s-1) & 1/3 \leq s \leq 2/3 \\ h(3s-2) & 2/3 \leq s \leq 1. \end{cases}$$

s =	0	1/3	2/3	1
$\emptyset(f)(s) =$ $(\rightarrow)$		h(0)=a	f(1)=a	h(1)=b
$(\leftarrow)$	h(1)=b	f(0)=a	h(0)=a	

Then $\emptyset(f)$ is continuous mapping of I into X with $\emptyset(f)(0) = \emptyset(f)(1) = b$.

We want to show that if f and g are homotopic with respect to a, then $\emptyset(f)$ and $\emptyset(g)$ are homotopic with respect to b. Let F be the continuous function establishing the homotopy between f and g such that $F(s, 0) = f(s)$ and $F(s, 1) = g(s)$ for all s in I and $F(0, t) = F(1, t) = a$ for all t in I. Then define

$$G(s, t) = \begin{cases} h(1-3s) & 0 \leq s \leq 1/3 & \text{for all } t \\ F(3s-1, t) & 1/3 \leq s \leq 2/3 & \text{for all } t \\ h(3s-2) & 2/3 \leq s \leq 1 & \text{for all } t \end{cases}$$

$s =$	0	$1/3$	$2/3$	1
$G(s, t) = (\rightarrow)$		$h(0)=a$	$F(1, t)=a$	$h(1)=b$
$(\leftarrow)$	$h(1)=b$	$F(0,t)=a$	$h(0)=a$	

G is continuous since it is continuous on each subinterval and well defined at the endpoints of the subintervals.

Then $G(s, 0) = \begin{cases} h(1-3s) & 0 \leq s \leq 1/3 \\ F(3s-1, 0) = f(3s-1) & 1/3 \leq s \leq 2/3 \\ h(3s-2) & 2/3 \leq s \leq 1 \end{cases} = \emptyset(f)(s)$

and $G(s, 1) = \begin{cases} h(1-3s) & 0 \leq s \leq 1/3 \\ F(3s-1, 1) = g(3s-1) & 1/3 \leq s \leq 2/3 \\ h(3s-2) & 2/3 \leq s \leq 1 \end{cases} = \emptyset(g)(s)$

$G(0, t) = h(1) = b = h(1) = G(1, t)$. Hence G verifies that $\emptyset(f) \underset{0,1}{\approx} \emptyset(g)$. This shows that $\emptyset$ maps a given homotopy class with respect to a into the same homotopy class with respect to b.

Now let g be any continuous mapping of I into X such that $g(0) = g(1) = b$. Define $\not\emptyset(g)$ by

$$\not\emptyset(g)(s) = \begin{cases} h(3s) & 0 \leq s \leq 1/3 \\ g(3s-1) & 1/3 \leq s \leq 2/3 \\ h(3-3s) & 2/3 \leq s \leq 1 \end{cases}$$

$s =$	0	$1/3$	$2/3$	1
$\not\emptyset(g)(s) = (\rightarrow)$		$h(1)=b$	$g(1)=b$	$h(0)=a$
$(\leftarrow)$	$h(0)=a$	$h(0)=b$	$h(1)=b$	

Then $\not\emptyset(g)$ is a continuous mapping of I into X with $\not\emptyset(g)(0) = \not\emptyset(g)(1) = a$.

Again, we want to show that if f and g are homotopic with respect to b, then $\not\emptyset(f)$ and $\not\emptyset(g)$ are homotopic with respect to a. Let

H be the continuous function establishing the homotopy between f and g such that H(s, 0) = f(s) and F(s, 0) = g(s) for all s in I and H(0, t) = H(1, t) = b for all t in I. Then define

$$J(s, t) = \begin{cases} h(3s) & 0 \leq s \leq 1/3 \quad \text{for all } t \\ H(3s-1, t) & 1/3 \leq s \leq 2/3 \quad \text{for all } t \\ h(3-3s) & 2/3 \leq s \leq 1 \quad \text{for all } t. \end{cases}$$

s =	0	1/3	2/3	1
J(s, t) = (→)		h(1)=b	H(1,t)=b	h(0)=a
(←)	h(0)=a	H(0,t)=b	h(1)=b	

J is continuous since it is continuous on each subinterval and well defined at the endpoints of the subintervals.

Then $J(s, 0) = \begin{cases} h(3s) & 0 \leq s \leq 1/3 \\ H(3s-1, 0) = f(3s-1) & 1/3 \leq s \leq 2/3 \\ h(3-3s) & 2/3 \leq s \leq 1 \end{cases} = ∅(f)(s)$

and $J(s, 1) = \begin{cases} h(3s) & 0 \leq s \leq 1/3 \\ H(3s-1, 1) = g(3s-1) & 1/3 \leq s \leq 2/3 \\ h(3-3s) & 2/3 \leq s \leq 1 \end{cases} = ∅(g)(s)$

J(0, t) = h(0) = a = h(0) = J(1, t). Hence J verified that ∅(f) $\approx_{0,1}$ ∅(g). This shows that ∅ maps a given homotopic class with respect to be into the same homotopic class with respect to a.

We would like to show that Ø and ∅ are inverses of each other. It will be sufficient to show that if f is a path such that f(0) = f(1) = a then ∅[Ø(f)] is homotopic to f with respect to a and if g is a path such that g(0) = g(1) = b then Ø[∅(g)] is homotopic to g with respect to b.

$$\bar{\emptyset}[\emptyset(f)](s) = \begin{cases} h(3s) & 0 \leq s \leq 1/3 \\ h(4-9s) & 1/3 \leq s \leq 4/9 \\ f(9s-4) & 4/9 \leq s \leq 5/9 \\ h(9s-5) & 5/9 \leq s \leq 2/3 \\ h(3-3s) & 2/3 \leq s \leq 1. \end{cases}$$

Define
$$F(s, t) = \begin{cases} h(3s) & 0 \leq s \leq (1/3)(1-t) \\ h(4-4t-9s) & (1/3)(1-t) \leq s \leq (4/9)(1-t) \\ f(9s-4+4t/8t+1) & (4/9)(1-t) \leq s \leq (1/9)(5+4t) \\ h(9s-5-4t) & (1/9)(5+4t) \leq s \leq (1/3)(t+2) \\ h(3-3s) & (1/3)(t+2) \leq s \leq 1. \end{cases}$$

Then
$$F(s, 0) = \begin{cases} h(3s) & 0 \leq s \leq 1/3 \\ h(4-9s) & 1/3 \leq s \leq 4/9 \\ f(9s-4/1) & 4/9 \leq s \leq 5/9 \\ h(9s-5) & 5/9 \leq s \leq 2/3 \\ h(3-3s) & 2/3 \leq s \leq 1 \end{cases} = \bar{\emptyset}[\emptyset(f)](s)$$

and
$$F(s, 1) = \begin{matrix} h(0) = a = f(0) & \\ f(s) & 0 \leq s \leq 1 \\ h(0) = a = f(1) & \end{matrix} = f(s)$$

$F(0, t) = h(0) = a = h(0) = F(1, t)$ so F is sufficient to establish $\bar{\emptyset}[\emptyset(f)] \underset{0,1}{\approx} f$.

$$\emptyset[\bar{\emptyset}(g)](s) = \begin{cases} h(1-3s) & 0 \leq s \leq 1/3 \\ h(9s-3) & 1/3 \leq s \leq 4/9 \\ g(9s-4) & 4/9 \leq s \leq 5/9 \\ h(6-9s) & 5/9 \leq s \leq 2/3 \\ h(3s-2) & 2/3 \leq s \leq 1. \end{cases}$$

$$\text{Define } G(s,\ t) = \begin{cases} h(1-3s) & 0 \le s \le (1/3)(1-t) \\ h(9s-3+4t) & (1/3)(1-t) \le s \le (4/9)(1-t) \\ g(9s-4+4t/8t+1) & (4/9)(1-t) \le s \le (1/9)(t+4t) \\ h(6-9s+4t) & (1/9)(5+4t) \le s \le (1/3)(t+2) \\ h(3s-2) & (1/3)(t+2) \le s \le 1. \end{cases}$$

$$\text{Then } G(s,\ 0) = \begin{cases} h(1-3s) & 0 \le s \le 1/3 \\ h(9s-3) & 1/3 \le s \le 4/9 \\ g(9s-4) & 4/9 \le s \le 5/9 \\ h(6-9s) & 5/9 \le s \le 2/3 \\ h(3s-2) & 2/3 \le s \le 1 \end{cases} = \phi[\bar{\phi}(g)](s)$$

$$\text{and } G(s,\ 1) = \begin{matrix} h(1) = b = g(0) \\ g(s) \quad 0 \le s \le 1 \\ h(1) = b = g(1) \end{matrix} = g(s)$$

$G(0,\ t) = h(1) = b = h(1) = G(1,\ t)$ so G is sufficient to establish $\phi[\bar{\phi}(g)] \underset{0,1}{\approx} g$.

Hence $\phi\colon \pi(X,\ a) \longrightarrow \pi(X,\ b)$ is one-to-one and onto.

To complete the isomorphism we need to verify that if f and g are two paths with $f(0) = g(0) = f(1) = g(1) = a$ then $\phi(f * g) \underset{0,1}{\approx} \phi(f) * \phi(g)$. For this purpose, define

$$H(s,\ t) = \begin{cases} h[1 - (6s/2-t)] & 0 \le s \le (1/6)(2-t) \\ f(6s+t-2) & (1/6)(2-t) \le s \le (1/6)(3-t) \\ h(6s+t-3) & (1/6)(3-t) \le s \le 1/2 \\ h(t+3-6s) & 1/2 \le s \le (1/6)(t+3) \\ g(6s-3-t) & (1/6)(t+3) \le s \le (1/6)(t+4) \\ h[6s - t - 4/2-t] & (1/6)(t+4) \le s \le 1 \end{cases}$$

Then
$$H(s, 0) = \begin{cases} h(1-3s) & 0 \leq s \leq 2/3 \\ f(6s-2) & 2/3 \leq s \leq 1/2 \\ h(6s-3) & 1/2 \leq s \leq 1/2 \\ h(3-6s) & 1/2 \leq s \leq 1/2 \\ g(6s-3) & 1/2 \leq s \leq 2/3 \\ h(3s-2) & 2/3 \leq s \leq 1 \end{cases}$$

$$= \begin{cases} h(1-3s) & 1 \leq s \leq 2/3 \\ f(6s-2) & 2/3 \leq s \leq 1/2 \\ g(6s-3) & 1/2 \leq s \leq 2/3 \\ h(3s-2) & 2/3 \leq s \leq 1 \end{cases} = \emptyset(f * g)$$

and
$$H(s, 1) = \begin{cases} h(1-6s) & 0 \leq s \leq 1/6 \\ f(6s-1) & 1/6 \leq s \leq 1/3 \\ h(6s-2) & 1/3 \leq s \leq 1/2 \\ h(4-6s) & 1/2 \leq s \leq 2/3 \\ g(6s-4) & 2/3 \leq s \leq 5/6 \\ h(6s-5) & 5/6 \leq s \leq 1 \end{cases} = \emptyset(f) * \emptyset(g)$$

$H(0, t) = h(1) = b = h(1) = H(1,t)$ so H is sufficient to show the desired homotopy.

DEFINITION FIVE: Let X, Y be topological spaces and let $f: X \longrightarrow Y$ be continuous and let a in X. Define $f': C_a(x)$ into $C_{f(a)}(Y)$ by $f'(h) = f \circ h$. And define $f_*: \pi(X, a)$ into $\pi(Y, f(a))$ by $f_*(\overline{h}) = \overline{f'(h)}$.

THEOREM FIVE: Show that f_* of definition five is well defined and a homomorphism of $\pi(X, a)$ into $\pi(Y, f(a))$.

Proof: If h in $C_a(X)$ and $h(0) = h(1) = a$, then $f'(h)(0) = f \circ h(0) = f(a)$ and $f'(h)(1) = f\ h(1) = f(a)$ so $f'(h)$ in $C_{f(a)}(Y)$.

To show f_* is well defined we need to show that $f_*(h) = \overline{f'(h)}$ is independent of the representative from the equivalence class $\overline{h}$. Suppose $h \approx_{0,1} k$, where $h,k: I \longrightarrow X$. Then there exists a continuous function $\emptyset: I \times I \longrightarrow X$ such that $\emptyset(t, 0) = h(t)$ and $\emptyset(t, 1) = k(t)$ and $\emptyset(0, t) = \emptyset(1, t) = a$. Define $\emptyset' = f \circ \emptyset: I \times I \longrightarrow Y$. Then $\emptyset'(t, 0) = f \circ \emptyset(t, 0) = f \circ h(t) = f'(h)$ and $\emptyset'(t, 1) = f \circ \emptyset(t, 1) = f \circ k(t) = f'(k)$. Since $\emptyset$ and f are continuous, so is $\emptyset'$. Thus $f'(h) \approx_{0,1} f'(k)$ and $f_*(\overline{h}) = \overline{f'(h)} = \overline{f'(k)} = f_*(\overline{k})$.

The group operation is "*" and by definition four $\overline{f} * \overline{g} = \overline{f * g}$ for f, g in $C_a(x)/\approx_{0,1}$. We need to show that $f_*(\overline{h} * \overline{k}) = f_*(\overline{h}) * f_*(\overline{k})$.

$$f_*(\overline{h} * \overline{k}) = f_*(\overline{h * k}) = \overline{f'(h * k)} = \overline{f\ (h * k)}$$
$$= \begin{cases} \overline{f \circ h}\ (2t) & 0 \leq t \leq 1/2 \\ \overline{f \circ k}\ (2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

$$f_*(\overline{h}) * f_*(\overline{k}) = \overline{f'(h)} * \overline{f'(k)} = \overline{f \circ h} * \overline{f \circ k}$$
$$= \begin{cases} \overline{f \circ h}\ (2t) & 0 \leq t \leq 1/2 \\ \overline{f \circ k}\ (2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Therefore $f_*(\overline{h} * \overline{k}) = f_*(\overline{h}) * f_*(\overline{k})$.

THEOREM SIX:[4] The fundamental group of the circle is isomorphic to the integers (under addition).

Proof: Let A be the equivalence class of f where $f(t) = (\cos 2\pi t, \sin 2\pi t)$ $0 \leq t \leq 1$. Let $g: I \longrightarrow S$ where $g(0) =$

$g(1) = (1, 0)$ be a closed path in S, the unit circle. First we must show that g belongs to the equivalence class A^m for some m in the integers. Let $U_1 = \{(x, y) \text{ in } S : y > -1/10\}$ and $U_2 = \{(x, y) \text{ in } S : y < 1/10\}$. Then U_1 and U_2 are connected open subsets of S and each is slightly larger than a semi-circle. U_1 union U_2 is S. U_1 and U_2 are each homeomorphic to an open interval of the real line, hence contractable. If g(I) in U_1 or g(I) in U_2, then g is equivalent to a constant path so g belongs to A^0. So assume that g(I) is not a subset of U_1 and g(I) is not a subset of U_2. Then partition [0, 1] such that (1) $0 = t_0 < t_1 < . . . < t_n = 1$, (2) $g[t_i, t_{i+1}]$ is a subset of either U_1 or U_2 for all i, $0 \leq i \leq n$, and (3) $g[t_{i-1}, t_i]$ and $g[t_i, t_{i+1}]$ are not both contained in the same open set U_1 or U_2. Let B be the equivalence class of path g and let B_i denote the equivalence class of $g/[t_{i-1}, t_i]$ for $1 \leq i \leq n$. Then $B = B_1 * B_2 * . . . * B_n$. Each B_i is path in U_1 or U_2. By (3) above $g(t_i)$ in both U_1 and U_2. U_1 intersect U_2 has two components, one containing the point (1, 0) and the other containing the point (-1, 0). For all i, $0 < i < n$, choose the path class $\emptyset_i$ in $U_1 \cap U_2$ with initial path $g(t_i)$ and terminal point (1, 0) or (-1, 0) depending on which component of $U_1 \cap U_2$ contains $g(t_i)$. Let $\emptyset_1 = B_1 * \emptyset_1$ and $\emptyset_i = \emptyset_{i-1}^{-1} * B_i * \emptyset_i$ for all i, $1 < i < n$ and $\emptyset_n = \emptyset_{n-1}^{-1} * B_n$. Then $B = \emptyset_1 * \emptyset_2 * . . . . * \emptyset_n$ where each $\emptyset_i$ is path class in U_1 or U_2 having its initial and terminal points in set $\{(1, 0)\ (-1, 0)\}$. For each i, if $\emptyset_i$ is a closed path class, $\emptyset_i = 1$ because U_1 and U_2 are simply connected. We may then assume any such $\emptyset_i$ has been dropped from the formula and changing notation if necessary, say $\emptyset_i$ for $1 \leq i \leq n$ are not closed paths. Since U_1 is simply connected, there exists a unique path N_1 in U_1 with initial point (1, 0) and terminal point (-1, 0). Also N_1^{-1} is a unique path in U_1 with initial point (-1, 0) and terminal point (1, 0). Simularly, there exists N_2, a unique path class in U_2 with initial point (-1, 0) and terminal point (1, 0). Then $N_1 N_2 = A$. For each index i, $\emptyset_i = N_1^{\pm 1}$ or $\emptyset_i = N_2^{\pm 1}$. Thus some cancellation is possible in $B = \emptyset_1 * \emptyset_2 * . . . * \emptyset_n$. After all possible pairs of terms have been cancelled, B is of one of the following forms:

$B = a$ (the constant function)

$B = N_1 * N_2 * N_1 * N_2 * N_1 * N_2 * . . . . * N_1 N_2$

$B = N_2^{-1} * N_1^{-1} * N_2^{-1} * N_1^{-1} * N_2^{-1} * N_1^{-1} \qquad N_2^{-1} * N_1^{-1}$

In the second case $B = A^m$ for some integer m larger than zero and in the third case $B = A^m$ for some integer m less than zero. Therefore, $B = A^m$ for m in integers in all cases and $\pi(S)$ is cyclic.

Secondly we must show that $\pi(S)$ is not finite. We will define the "degree" of a closed path in $\pi(S)$ as an integer denoting the number of times the path wraps around the circle. Consider S as the unit circle in the complex plane. Then S is a group under multiplication. If z in S, denote by a(z) the angle of (z), ie the angle in radians from the positive real axis to the segment joining 0 and z. Hence for z in S, a(z) in reals but not unique. However, $a(z_1) + a(z_2) = a(z_1 z_2)$ and $a(z_1) - a(z_2) = a(z_1/z_2)$. Let $h: I \longrightarrow S$ be a closed

path with $h(0) = h(1) = 1$. Choose partition of I such that $0 = t_0 < t_1 < \ldots\ldots < t_n = 1$, and if t and t' in $[t_{i-1}, t_i]$ then $/h(t) - h(t')/ < 1$. (The uniform continuity of h will guarantee the existance of such a partition.) For all index i, $0 \leq i \leq n$, let $\emptyset_i$ be the unique determination of angle $h(t_i)/h(t_{i-1})$ such that $-\pi/2 < \emptyset_i < \pi/2$. Then define degree $h = (1/2\pi)\sum_{i=1}^{m} \emptyset_i$. Then h is an integer since $\sum_{i=1}^{m} \emptyset_i$ is a determination of the angle of the complex number $\prod_{i=1}^{m} h(t_i)/h(t_{i-1}) = h(t_n)/h(t_0) = 1$.
Since any two subdivisions of I into subintervals have a common refinement, consider a refinement of a given subdivision.
Subdivide the subintervals $[t_{i-1}, t_i]$ by a point s_i such that $t_{i-1} < s_i < t_i$ and replace $\emptyset$ by $\emptyset_i'$ and $\emptyset_i''$ where $\emptyset_i' = a[h(s)/h(t_{i-1})]$ and $\emptyset_i'' = a[h(t_i)/h(s)]$, where $\emptyset_i'$ and $\emptyset_i''$ are both in open interval from $-\pi/2$ to $\pi/2$. Consequently $\emptyset_i = \emptyset_i' + \emptyset_i''$. Hence the definition of the degree of h is independent of the choice of subdivisions.
Suppose that h is homotopic to g. Then there exists a continuous function $F: I \times I \rightarrow S$ such that $F(t, 0) = h(t)$, $F(t, 1) = g(t)$ and $F(0, s) = F(1, s) = 1$. Choose partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ and $0 = s_0 < s_1 < \ldots\ldots < s_m = 1$ such that F maps each rectangle $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$ into a subset of S with diameter less than 1. Let $\emptyset_i' = a[F(t_i, s_{j-1}) / F(t_{i-1}, s_{j-1})]$ and

$\emptyset_i'' = a[F(t_i, s_j) / F(t_{i-1}, s_j)]$ where $/\emptyset_i'/ < \pi/2$ and $/\emptyset_i''/ < \pi/2$. Let $\psi_i = a[F(t_i, s_j) / F(t_i, s_{j-1})]$, $/\psi_i/ < \pi/2$ for all i = 0, 1, ... n. $\emptyset_i'' - \emptyset_i'$ and $\psi_i - \psi_{i-1}$ are both determinations of the angle of the complex number $f(t_i s_j) \cdot F(t_{i-1}, s_{j-1}) / F(t_{i-1} s_j) \cdot F(t_i\ s_{j-1})$, so they differ by a multiple of 2π. However, the restrictions on the absolute value guarantee that $\phi_i'' - \phi_i' = \phi_i - \phi_{i-1}$. Then $\sum_{i=1}^{m} \phi_i'' - \sum_{i=1}^{m} \phi_i' = \sum_{i=1}^{m} (\phi_i - \phi_{i-1}) = \phi_n - \phi_0 = 0$. Therefore $\sum_{i=1}^{m} \phi_i'' = \sum_{i=1}^{m} \phi_i'$. Hence degree h = degree g. Now to any element B in $\pi(S)$ we can assign a unique integer, the degree of B. For any integer m, $h_m : I \to S$ defined by $h_m(t) = \cos 2m\pi t + i \sin 2m\pi t$ has degree m. Thus $\pi(S)$ has infinite order.

THEOREM SEVEN: Let S be a circle in R_3. Then $\pi(R_3 - S)$ is isomorphic to the integers under addition.

Proof: We need to show that (1) if h in $C_a(R_3 - S)$ then $h \underset{0,1}{\approx} f^n$ for some n in the integers, and (2) $f^n \underset{0,1}{\approx} f^m$ iff n = m.

LEMMA: $R_3 - S$ is arc-wise connected.

For $x = (x_1, x_2, x_3)$ in R_3 - (YZ plane) define $g(x, t) = ([1-t]x_1 + t, [1-t]x_2, [1-t]x_3)$. Then $g(x, 0) = x$ and $g(x, 1) = (1, 0, 0) = a$. Then $g[0, 1]$ is a subset of $R_3 - S$ and g is continuous.

For $x = (0, x_2, x_3)$ in $R_3 - S$ intersected with the YZ plane define

$$h(x, t) = \begin{cases} (2t, x_2, x_3) & 0 \leq t \leq 1/2 \\ (1, 2[1-t]x_2, 2[1-t]x_3) & 1/2 \leq t \leq 1 \end{cases}$$

Then $h(x, 0) = x$ and $h(x, 1) = (1, 0, 0)$. Hence $h[0, 1]$ is a subset of $R_3 - S$ and h is continuous.

Therefore, if x in $R_3 - S$, there exists a continuous path connecting x to (1, 0, 0) so $R_3 - S$ is arc-wise connected.

Returning to the proof of the theorem, we may fix S, a and f without loss of generality. Let $S = \{(0, y, z): (y+5/4)^2 + z^2 = 1\}$, $f^0 = (1, 0, 0) = a$, $f^1(t) = \{(\cos 2\pi t, \sin 2\pi t, 0): 0 \leq t \leq 1\}$,

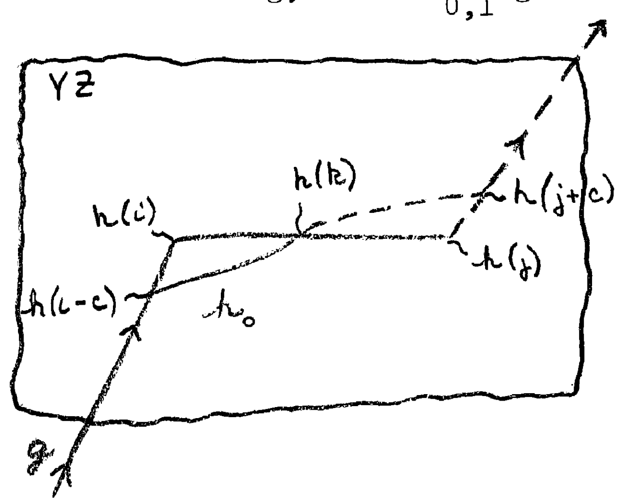
$f^{-1}(t) = f(1-t)$, $f^2 = f * f$, and $f^k(t)$ travel path of f k times with $f^k(i/k) = a$ for $i = 0, 1, 2, \ldots, k$.

(1) Suppose h in $C_a(R_3 - S)$. Let U be an open cover of h(I) composed of open spheres in R_3 which do not intersect S. Since I is a compact set and h is a continuous function, h(I) is also compact. Let V be a finite subcover of U, ie $V = \{U_i : U_i \text{ in } U, i = 1, 2, \ldots, n\}$ and h(I) a subset of $\bigcup V$. Partition I by selecting p_j in [0, 1] such that $0 = p_0 < p_1 < \ldots . < p_k = 1$ and $h[p_j, p_{j+1}]$ is contained in one of the elements of V for all $j = 0, 1, \ldots, k-1$. Since the set I is compact, this can be done with a finite number of points p_j. Define g to be the continuous polygonal path joining successive $h(p_j)$ by

for t in $[p_i, p_{i+1}]$, $g(t) = \dfrac{p_{i+1} - t}{p_{i+1} - p_i} h(p_i) + \dfrac{t - p_i}{p_{i+1} - p_i} h(p_{i+1})$.

For each interval $I_i = [p_i, p_{i+1}]$ we have specified $h(I_i)$ be contained in one of the elements of V so $h(I_i)$ is contained in one of the open spheres in U. By the convexity of open spheres, $g(I_i)$ is contained in this same open sphere. Since this sphere does not intersect S, $h(I_i)$ can be continuously deformed onto $g(I_i)$. Hence h can be continuously deformed onto g, ie $h \underset{0,1}{\approx} g$.

If g(t) lies in the YZ plane for all t in [i, j] with $0 < i < j < 1$, choose c such that g(t) does not lie in YZ plane for all t in [i-c, i) and t in (j, j+c]. Fix k in [i, j] and define $h_0(t)$.



$$h_0(t) = \begin{cases} \dfrac{k - t}{k - i + c}\, g(i - c) + \dfrac{t - i + c}{k - i + c}\, g(k) & i-c \le t \le k \\[2ex] \dfrac{j + c - t}{j + c - k}\, g(k) + \dfrac{t - k}{j + c - k}\, g(j + c) & k \le t \le j+c \end{cases}$$

Since g did not cross S in [i, j], h_0 will be continuous and g can be continuously deformed onto h_0. Hence $h_0 \approx_{0,1} g$. But $g \approx_{0,1} h$ so $h_0 \approx_{0,1} h$. Since h_0 is a polygonal path composed of a finite number of straight line segments, none of which lie in the YZ plane, h_0 can intersect the YZ plane only a finite number of times.

Let $\{t_i\}$ be the values of t in [0, 1] such that $h(t_i) = (0, y, z)$ with $t_1 < t_2 < \ldots < t_n$. For each pair t_i and t_{i+1} satisfying

(I) at t_i h_0 goes from the positive x side of the YZ plane to negative x and at t_{i+1} h_0 goes from negative x to positive x, or vise versa and

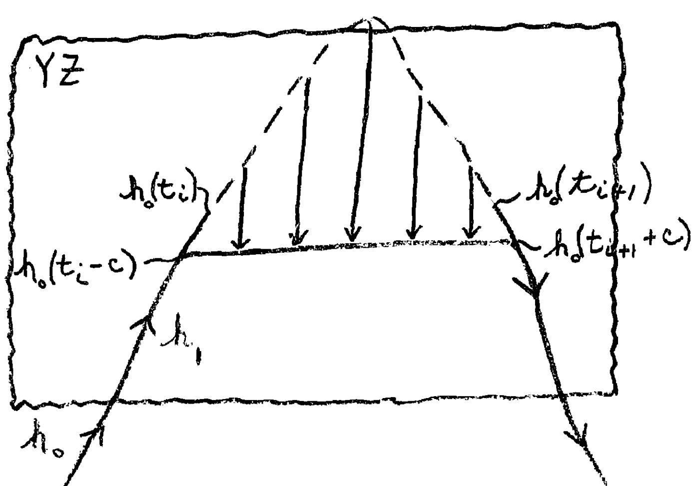
(II) $h_0(t_i)$ and $h_0(t_{i+1})$ are both interior to S or both exterior to S,

choose c such that $t_{i-1} < t_i - c < t_i < t_{i+1} < t_{i+1} + c < t_{i+2}$

Define $h_1 : [t_i - c, t_{i+1} + c] \longrightarrow R_3 - S$ by

$$h_1(t) = \frac{t_{i+1} + c - t}{t_{i+1} - t_i + 2c}\, h(t_i - c) + \frac{t - t_i + c}{t_{i+1} - t_i + 2c}\, h(t_{i+1} + c)$$

Then $h_1[t_i - c, t_{i+1} + c]$ lies entirely on one side of the YZ plane and is continuous on this interval. Define $h_1 = h_0$ otherwise. Since h_0 does not loop around S in this interval



(by condition II on t_i and t_{i+1}) h_1 can be continuously deformed onto h_0. Hence $h_1 \approx_{0,1} h_0 \approx_{0,1} h$.

Now renumber the values of t where $h_1(t) = (0, y, z)$ as $t_{1'}, t_{2'}, \ldots t_{n'}$ where $t_{1'} < t_{2'} < \ldots . < t_{n'}$ and define h_2 simular to h_1. Again $h_2 \approx_{0,1} h_1 \approx_{0,1} h$. Repeat this process as many times as necessary until the $f(t_i)$ are alternately interior and exterior to S. Call the final function h'. By successive transitivity $h' \approx_{0,1} h$. Also h' either always crosses the YZ plane from positive x to negative x or always crosses the YZ plane from negative x to positive x.

Let $T_{h'} = \{t_1, t_2, \ldots t_n\}$ be the set of all t's where h'(t) is on the YZ plane internal to S.

Define $\emptyset(t_i) = \begin{cases} 1 & \text{if } h' \text{ is going from negative x to positive x at } t_i \\ -1 & \text{if } h' \text{ is going from positive x to negative x at } t_i. \end{cases}$

Define $W(h) = \sum_{t_i \in T_{h'}} \emptyset(t_i)$.

W is then defined for each h in $C_a(X)$ and W(h) is an integer.

IF W(h) = 0:

Let h' be the function associated with h by the above process. Then h'[0, 1] lies entirely on the positive side of the YZ plane. This half space has no deleted points so h' is contractable to h'(0) = h'(1) = (1, 0, 0) = a. Hence $h \approx_{0,1} h' \approx_{0,1} a = f^0$.

IF W(h) = 1

Let h' be the function associated with h by the above process.

Then h' pierces the YZ plane internal to S once, say at $t = k$.

Define $h''(t) = \begin{cases} h'[(4/3)kt] & 0 \leq t \leq 3/4 \\ h'[4(1-k)t - 3 + 4k] & 3/4 \leq t \leq 1 \end{cases}$

Then h" follows the same path as h' only varying the rate of travel. Hence $h'' \underset{0,1}{\approx} h' \underset{0,1}{\approx} h$. Both h" and f^1 pierce the YZ plane internal to S at $t = 3/4$. Define $\emptyset: I \times I \longrightarrow R_3 - S$ by $\emptyset(t, t'') = (1-t'')h''(t) + t''f(t)$. Ø will be continuous since for a fixed t, Ø defines a straight line connecting h"(t) and f'(t) and this line never passes thru S. Therefore $h'' \underset{0,1}{\approx} f^1$ and so $h \underset{0,1}{\approx} f^1$.

IF $W(h) = k > 0$

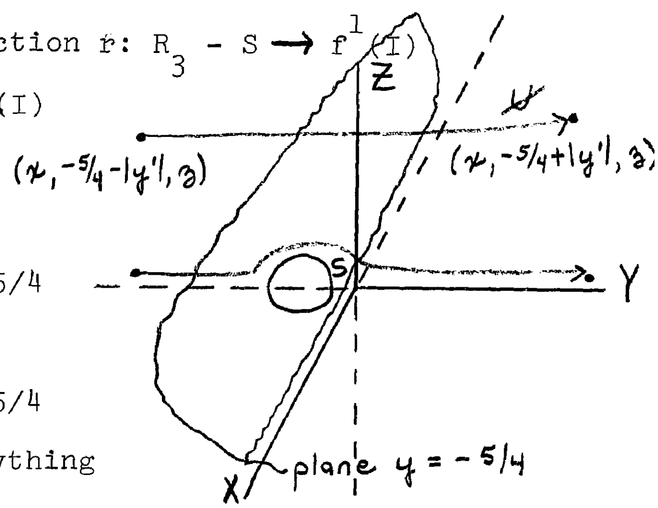
The associated function h' crosses the YZ plane internal to S from negative x to positive x k times, say at $t_1, t_2, \ldots, t_k$. Define h"(t) simular to the case $W(h) = 1$ so that h"(t) crosses YZ internal to S at values $t = (1/k)[(3/4) + i]$ for $i = 1, 0, 2..$ $k-1$. Then $h'' \underset{0,1}{\approx} h'$ since we have again merely changed the rate of travel. Now deform h" onto h"' where h"' = h" except on intervals $[(i/k) - (1/8k), (i/k) + (1/8k)]$ for $i = 1, 2, ..$ $k-1$. and $h'''(i/k) = (1, 0, 0) = a$ for $i = 1, 2, \ldots k-1$ by inserting continuous arcs from $h''[(i/k) - (1/8k)]$ to $(1, 0, 0)$ to $h''[(i/k) + (1/8k)]$. The arc-wise connectedness of $R_3 - S$ assures that this can be done. Hence $h''' \underset{0,1}{\approx} h'' \underset{0,1}{\approx} h$. Define $\emptyset(t, t''): I \times I \longrightarrow R_3 - S$ by $\emptyset(t, t'') = (1-t'')h'''(t) + t''f^k(t)$. If $t = i/k$ $i = 0, 1, \ldots, k$, $\emptyset(t, t'') = (1, 0, 0) = a$ for all t" since $h'''(t) = a = f^k(t)$. And Ø is continuous for fixed t since Ø defines straight lines connecting h"'(t) and $f^k(t)$ and this line never passes thru S. Therefore $h''' \underset{0,1}{\approx} f^k$

and since $h''' \underset{0,1}{\approx} h$, $h \underset{0,1}{\approx} f^k$.

IF $W(h) = k < 0$

h can be shown homotopic to f^k by argument simular to that for $k > 0$.

We have now defined the function W(h) with values in the integers such that for each h in $C_a(X)$ if $W(h) = k$ then $h \underset{0,1}{\approx} f^k$.

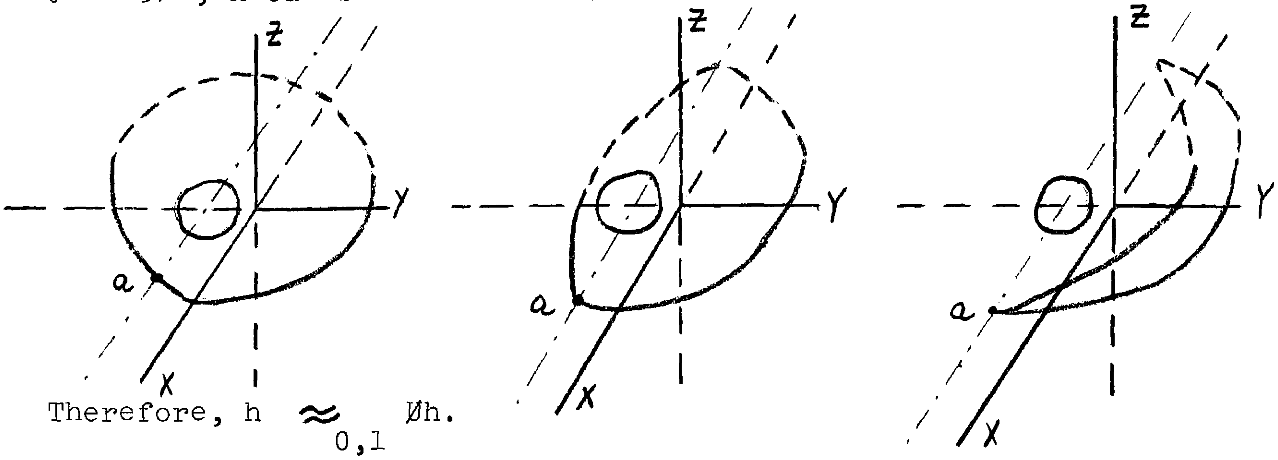


(2) We first seek to define a function $r: R_3 - S \rightarrow f^1(I)$ such that $r(x) = x$ for all x in $f^1(I)$

Plane $y = -5/4$ contains the center of S. Define $Ø: R_3 - S \rightarrow R_3 - S$ by

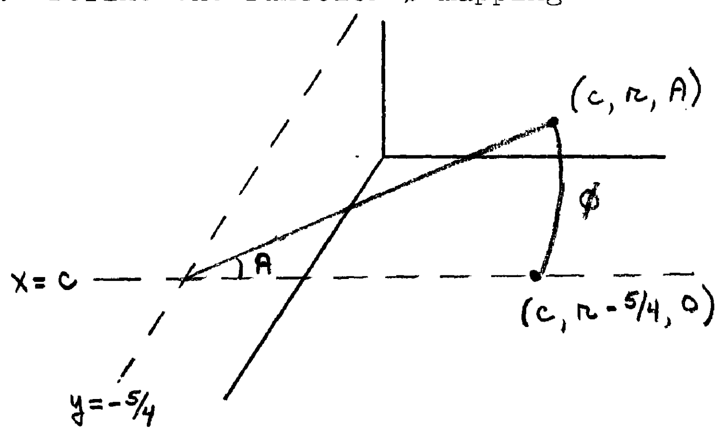
$$Ø(x, y, z) = \begin{cases} (x, y, z) & \text{if } y \geq -5/4 \\ (x, /y/-5/4, z) & \text{if } y < -5/4 \end{cases}$$

Then Ø is continuous and maps everything on the left of the plane $y = -5/4$ to the point on the right of this plane and (x, r, A) maps to $(x, r, \pi - A)$ in polar form. No points are mapped to S due to the symmetry of the circle. If h(I) does not intersect the YZ plane in the region $-1 \leq z \leq 1$, h(I) can be continuously deformed into Øh(I) by mapping each point with $y < -5/4$ through the straight line segment (x, y, z) to $(x, -y-5/4, z)$. If h(I) does intersect the YZ plane in the region $-1 \leq z \leq 1$ and $y < -5/4$ in a segment $h[t_1, t_2]$ this segment will have to be stretched our around S rather than follow a straight line segment. By stretching each such section around S simultaneously to passing other sections past S, then shrinking it back to the reflection of h across the plane $y = -5/4$, h can be continuously deformed onto $\emptyset h(I)$.

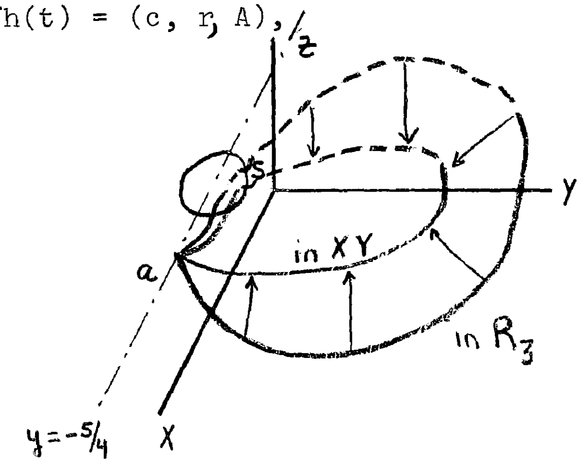


Therefore, $h \underset{0,1}{\approx} \emptyset h$.

For each plane $x = c$, the point (c, y, z) has polar representation (c, r, A) with respect to the ray $\overrightarrow{(c, -5/4, 0)\ (c, y, z)}$ and A measured in the plane $x = c$. Define the function Ø mapping the half space $y \geq -5/4$ into the XY plane by $\emptyset(c, r, A) = (c, r - 5/4, 0)$. Then Ø is continuous and folds the half space $y \geq -5/4$ onto the half plane $y \geq -5/4$ of the XY plane.

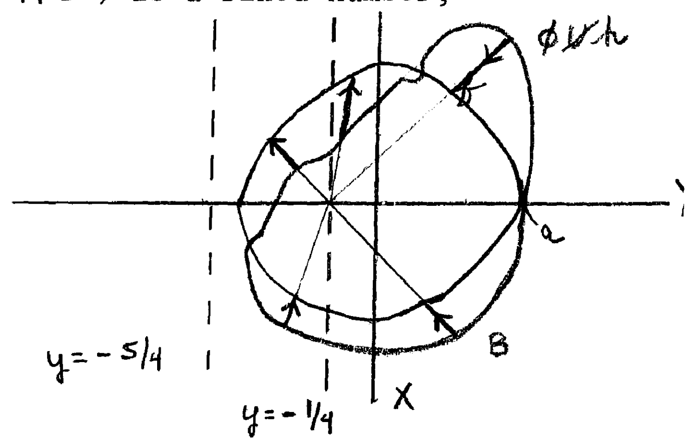


Nothing is mapped to $(0, -1/4, 0)$ since $-1/4 = 1 - 5/4$ and 1 is the radius of S. Define T such that if $\emptyset h(t) = (c, r, A)$, $T(t, t'') = (c, r - 5/4\, t'', (1-t'')A)$. Then $T(t, 0) = (c, r, A)$, $T(t, 1) = (c, r-5/4, 0)$, and T is continuous in t" for fixed t. Since T never crosses S, it will also be continuous in t for fixed t". Hence $\emptyset h(I)$ can



be continuously deformed onto Ø∅h(I). Therefore ∅h $\underset{0,1}{\approx}$ Ø∅h.

Each point in the XY plane (x, y, 0) has polar representation (r', A') with respect to the ray $\overrightarrow{(0, -1/4, 0)\ (x, y, 0)}$ and A' measured in the XY plane. Define function E mapping the XY plane - $\{\{(x, y, z) : y < -5/4\} \cup (0, -1/4, 0)\}$ into f^1, by E(r', A') = $\overrightarrow{A'} \cap f^1$, $r' > 0$. Then E is continuous and onto f^1. On each ray $\vec{A}$ eminating from (0, -1/4, 0), the distance from (0, -1/4, 0) to ($\overrightarrow{A} \cap f'$) is a fixed number, say r_A. Define function B by if Ø∅h(t) = (r', A'), then B(t, t") = (1-t")r' + t"r_A,A'. Then B is continuous in t' by linearity and since B never crosses S, it will also be continuous in t for fixed t".



Hence Ø∅h(I) can be continuously deformed into EØ∅h(I). Hence Ø∅h(I) $\underset{0,1}{\approx}$ EØ∅h(I).

Let r = E∘Ø∘∅. Then r: R_3 - S $\longrightarrow f^1$(I) is continuous and onto. We have shown that h $\underset{0,1}{\approx}$ EØ∅h = rh.

The range of rh(I) is a power of f. If W(h) = k, h $\underset{0,1}{\approx}$ f^k and rh $\underset{0,1}{\approx}$ f^k where f^k is in the space f(I). Hence $f^m = f^n$ iff m = n by theorem six.

FOOTNOTES

1 All of the definitions are taken from Babcock, William W., Supplementary Notes.

2 This proof is based on that given by Hocking and Young, pages 160 - 165.

3 This proof is based on that given by Wallace, pages 85 - 88.

4 This proof is based on that given by Massey, pages 68 - 73.

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