9-2014

Cold Hard Fractals

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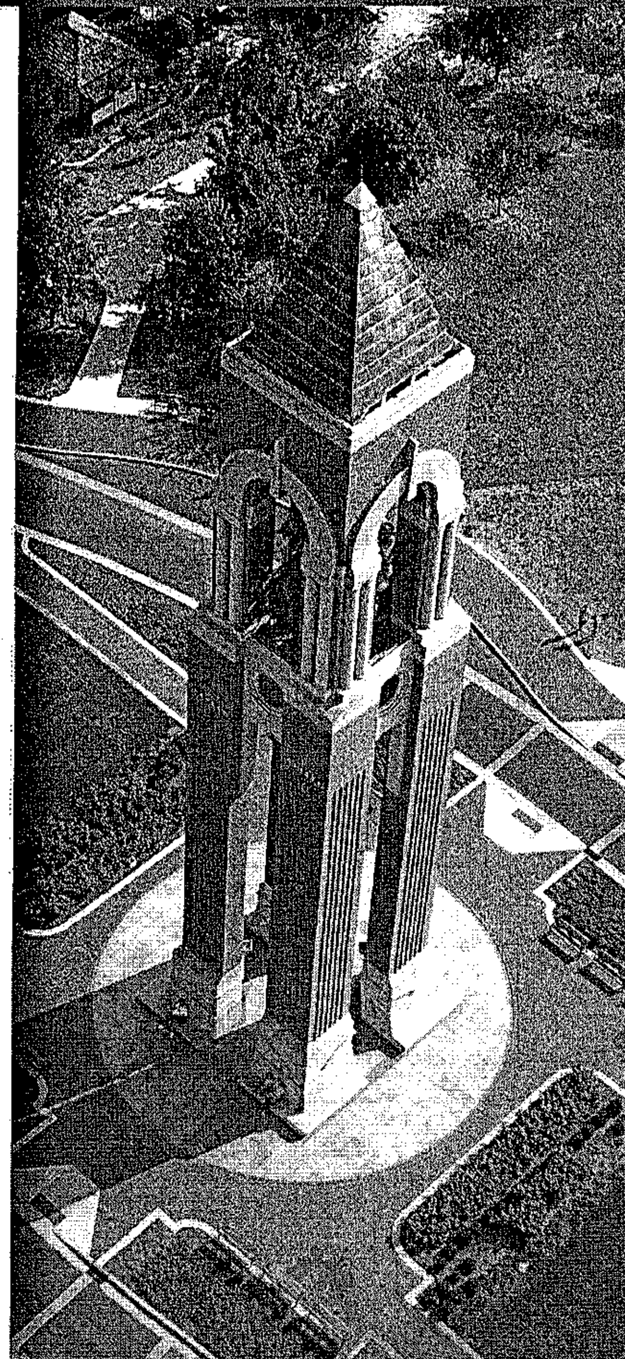
In Michigan's Upper Peninsula, the length of winter is not counted by the days it snows or by the days below zero, but rather the days a winter coat is required to venture outdoors. This past winter by all measures was long and cold. The winter of 2013-2014 in the Upper Peninsula saw a string of 75 consecutive days where the mercury in the thermometer did not rise above the freezing mark (http://www.crh.noaa.gov). This exceptionally cold winter was admired and feared by the residents of the Great Lakes Region. Arguably the most admired feature of the winter was located in the Apostle Islands National Lakeshore in northern Wisconsin. The ice caves of the Apostle Islands are truly a work of art by old man winter, which after failing to form for the past six years drew record crowds of well over 100,000 for the winter (https://www.facebook.com/apostleislandsnationallakeshore).

The Polar Vortex this past winter yielded some remarkable recreational opportunities on Lake Superior. Not only did nearly all of Lake Superior freeze completely over, it was frozen over for most of the winter. With this accessibility, sandstone cliffs on the shores of Lake Superior at other places besides the Apostle Islands such as the Grand Island National Recreation Area and the Pictured Rocks National Lakeshore provided back-country ski and snowshoe opportunities to ice caves which are normally not accessible in the winter. After several trips (and many pictures), one of the authors, who at the time was teaching a mathematics modeling course for pre-service secondary mathematics teachers, became curious whether a classification scheme could be used to classify the "jaggedness" of the ice formations in the caves and whether this could be incorporated into the modeling course.

The classic question many mathematics teachers dread to hear their students utter is, "When will we ever use this stuff?" This comment is not unique to the high school curriculum. It is also stated quite frequently in college mathematics courses. Although many college mathematics students enjoy studying fractal geometry, more times than not, they question whether the content can be useful outside the classroom. This article describes a mathematical investigation that applied content from a fractal geometry unit to the real-word phenomena of ice formations on Lake Superior created by the recent winter.

Students in the course were given the task of determining the fractal dimension of various ice formations chosen from a database of pictures located at http://euclid.nmu.edu/~dbuhl/Ice/pics.html. Wikipedia (and other Internet sites) have a dearth of information regarding fractal dimension. Fractal dimension can be defined as a ratio providing a statistical index of complexity by comparing how detail in a pattern (strictly speaking, a fractal pattern) changes with the scale at which it is measured (http://en.wikipedia.org/wiki/Fractal_dimension). Prior to the class investigation into the fractal dimension of Lake Superior ice formations, the class studied the fractal dimension of popular fractals including the Cantor Set, Koch's Snowflake, and Sierpinski's Triangle. For a thorough listing of the fractal dimension of some of the more "popular" fractals, the authors refer the reader to http://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension.

Fractals can be classified into two different types: self-similar and non self-similar. Self-similar fractals look roughly the same on any scale. The Cantor Set, Koch's Snowflake, and Sierpinski's Triangle are examples of self-similar fractals. Coastlines and ice formations are examples of non self-



similar fractals. These fractals differ in complexity when you change the scale.

There are several different algorithms one can use to determine the fractal dimension of both self-similar and non-similar objects. The box counting algorithm was used in this activity to find the fractal dimension of the ice formations. The box counting method for finding fractal dimensions is based on determining the mathematical relationship between the size of a "box" and the number of "boxes" need to cover an object. To illustrate, we will calculate the fractal dimension of the face of a sheet of a paper which should equal 2 since it is two dimensional and it covers the entire sheet of paper. For simplicity's sake, we will assume we have a square sheet of paper. Our first "box" will be a square and we will initially start with the size of square equal to the square sheet of paper. For notation purposes, r represents scale size of "box" or square in this case and N(r) represents the number of "boxes" or squares to cover the object (Diagram 1). For our example, r =1 and N(r) = 1.

Now suppose the dimensions of the "box" are 1/2 the original size. We then would need 4 "boxes" to cover the piece of paper. Thus, r = 1/2 and N(1/2) = 4. Continuing with this process, if we use "boxes" 1/4 the size of the original "box" we would need 16 "boxes" to cover the original sheet of paper. For this case, r = 1/4 and N(1/4) = 16. As r changes, a candidate for the relationship between r and N(r) could be $(1/r)^2 = N(r)$. If this is the case and we know the dimension of the face of the square sheet of paper is 2, we could make the conclusion that the relationship $(1/r)^D = N(r)$ should yield us dimension D. The authors leave it to the reader to verify the relationship for D is as follows:

$$D = \frac{\log(N(r))}{\log r}$$

For self-similar fractals, the value for D will not change at any scale since at any scale the relationship between size of the box and the number of boxes remain constant. This is not the case for non self-similar fractals. Instead, we consider the rate of change (or

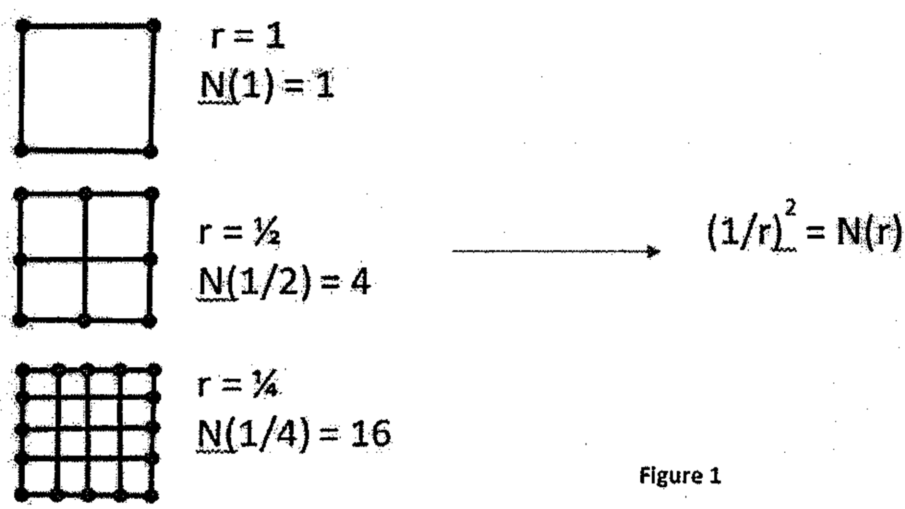


Figure 1

slope) in the relationship between r and N(r) at various scales. This relationship for D can be derived as:

$$D = \frac{\log N(a) - \log N(b)}{\log(\frac{1}{S_a}) - \log(\frac{1}{S_b})}$$

where N(a) is the number of boxes counted when S_a is the size of boxes and N(b) is the number of boxes counted when S_b is the size of the boxes. For a thorough explanation of the box counting the method the author refers the reader to http://fractalfoundation.org/OFC/OFC-12-4.html. To further illustrate this method the authors have provided two examples of students' work in calculating fractal dimension of ice formations as well as an explanation of the process used to complete this activity.

Diagram 4 illustrates Gabe's work for finding the fractal dimension of the ice formation in Diagram 2. First, Gabe traced the ice formation onto graph paper. Anticipating having to use three different sizes of boxes, Gabe defined his unit one box as a 2x2 box in terms of the predetermined boxes on the graph paper. Gabe then counted, N(1) = 43 representing the number of boxes (43) "touched" by ice in the 2x2 boxes. Similarly, N(1/2) = 113 represented the number of boxes (113) "touched" by ice in 1x1 boxes and N(1/4) = 283 represented the number of boxes (283) "touched" by ice in 1/2 x 1/2 sized boxes. With these three data points {(1, 43),(1/2,113), (1/4, 283)} Gabe then calculated the slope between points (1/2, 113) and (1,43) which yielded 1.393 and the slope between the points (1/4,283) and (1/2,113) which yielded 1.324.

Finally, Gabe calculated the average of these two slopes to arrive at fractal dimension of D=1.359. A dimension of 1.359 makes intuitive sense here because the outline is somewhere between a straight line (D=1) and a plane (D=2).

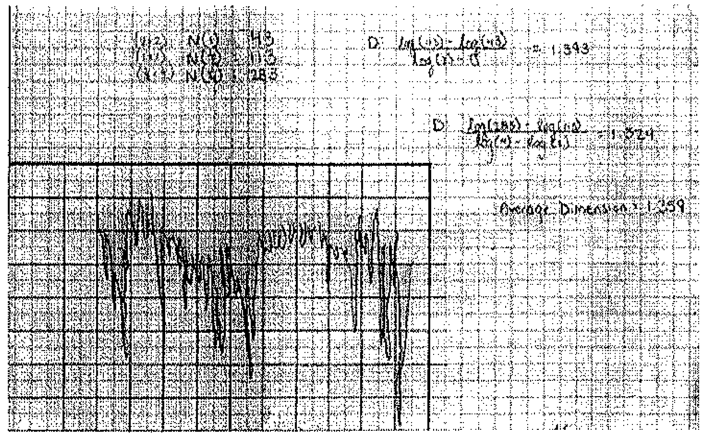


Figure 4: Gabe's Fractal Dimension Calculations

Diagram 5 shows how Charlie analyzed the fractal dimension of the ice formation in Diagram 2. Charlie chose to trace the ice formation with a fine tipped marker to make the outline easier to trace. Then pressing the picture of the ice formation with graph paper on top up against a window pane Charlie traced the ice formation onto the graph paper. Similar to the Gabe's work above, Charlie sectioned off the graph paper into 2x2 boxes to be the unit box. When counting the smaller boxes, Charlie counted the boxes in a column and then tallied the number of boxes in each column. After completing his count Charlie then summed these numbers to arrive at a total. This method can be more efficient when dealing with a larger sum of boxes and also aids in the accuracy of the count because the student does not have to keep track of a large number as they count the boxes. Diagram 5 shows that Charlie calculated his "points" to be {(1,104), (1/2,263) and (1/4,607)}. Then using the formula above, he calculated the slope between all three points resulting in dimensions of 1.207, 1.273 and 1.338. These results better illustrate why it is important to calculate the average of the three slopes because it is evident that if either the lower of higher value of this set was excluded then the final dimension would be skewed. It is also noteworthy to point out that the slope between N(1/4) and N(1) in

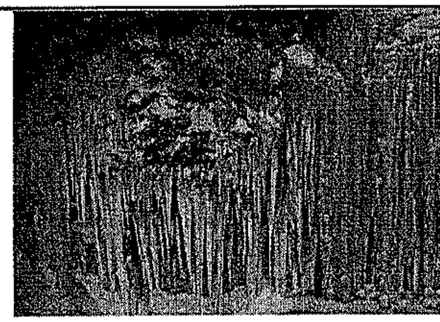


Figure 2: Gabe's Ice Formation

Figure 3: Charlie's Ice Formation

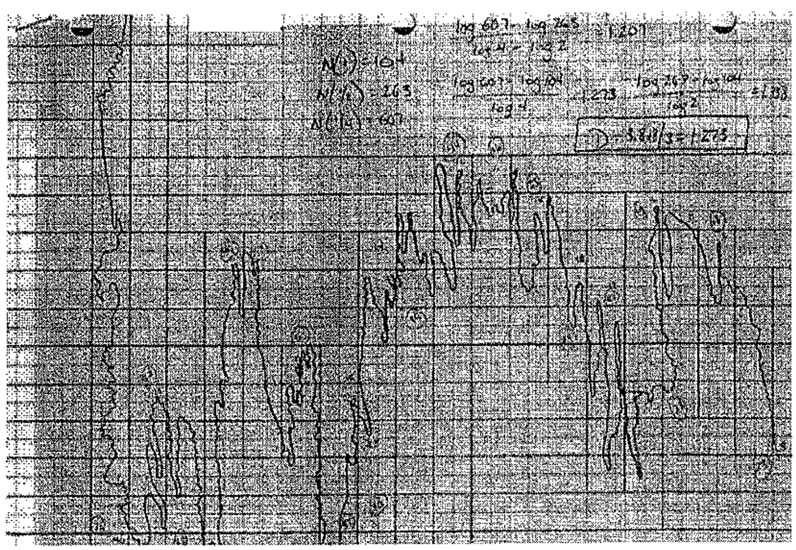


Figure 5: Charlie's Fractal Dimension Calculations

Diagram 5 is close to the average of all the slopes, which holds to be true for the calculations in Diagram 4 as well ($D = \frac{\log(283) - \log(43)}{\log(4)} = 1.359$). This should appeal to our intuition because the slopes between the other two points are the between the two points whose slope closest to the average dimension.

Diagram 5 shows how Charlie analyzed the fractal dimension of the ice formation in Diagram 2. Charlie chose to trace the ice formation with a fine tipped marker to make the outline easier to trace. Then pressing the picture of the ice formation with graph paper on top up against a window pane Charlie traced the ice formation onto the graph paper. Similar to the Gabe's work above, Charlie sectioned off the graph paper into 2x2 boxes to be the unit box. When counting the smaller boxes, Charlie counted the boxes in a column and then tallied the number of boxes in each column. After completing his count Charlie then summed these numbers to arrive at a total. This method can be more efficient when dealing with a larger sum of boxes and also aids in the accuracy of the count because the student does not have to keep track of a large number as they count the boxes. Diagram 5 shows that Charlie calculated his "points" to be {(1,104), (1/2,263) and (1/4,607)}. Then using the formula above, he calculated the slope between all three points resulting in dimensions of 1.207, 1.273 and 1.338. These results better illustrate why it is important to calculate the average of the three slopes because it is evident that if either the lower of higher value of this set was excluded then the final dimension would be skewed. It is also noteworthy to point out that the slope between N(1/4) and N (1) in Diagram 5 is close to the average of all the slopes, which holds to be true for the calculations in Diagram 4 as well ($D = \frac{\log(283) - \log(43)}{\log(4)} = 1.359$). This should appeal to our intuition because the slopes between the other two points are the between the two points whose slope closest to the average dimension.

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Figure 6: Practice!